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Stationary distributions of dust and electromagnetic fields in general relativity

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Abstract. In the present paper we give a number of solutions corresponding to a stationary distribution of rotating charged dust with vanishing Lorentz force. It is found that in all but one case there are closed time-like lines. Further, the paper shows the existence of some homogeneous stationary universes other than that of Gödel and Einstein when, besides the matter field, there is an electromagnetic field as well.

1. Introduction

Rotation in general relativity has some rather ill-understood aspects. One such feature is the occurrence of closed time-like lines which imply a violation of causality. The well-known solutions of Van Stockum (1937), Gödel (1949) and Wright (1965) and the recently discovered rotating charged-dust solution of Som and Raychaudhuri (1968) all possess this peculiar feature. However, the stationary solution of Maitra (1966), which is a case of non-rigid rotation (as there is a shear), does not have any closed time-like lines.

Again, it is well known that the only stationary homogeneous dust-filled cosmological solutions are the Gödel and Einstein universes. In the present paper we give a number of solutions corresponding to stationary distributions of rotating charged dust in which the Lorentz force vanishes. It is found that in all but one case there are closed time-like lines. Further, there occur some stationary homogeneous universes where, along with the uniform dust distribution, there is an electromagnetic energy distribution.

The solutions that are presented have the following characteristics:

(i) The solutions are all stationary, so that the expansion vanishes, and excepting in one case the shear also vanishes.

(ii) The matter is in the form of incoherent dust. However, as in the Gödel solution, a group of solutions with non-vanishing cosmological constant can alternatively be interpreted as solutions for a perfect fluid with non-vanishing pressure and vanishing λ.

(iii) Besides the matter field there is an electromagnetic field as well, and the dust, except in one case, is electrically charged. However, in all the cases considered the Lorentz force is assumed to vanish, so that the world lines of the dust are always geodesics.

(iv) The solutions, except in one case, exhibit cylindrical symmetry and some of the solutions are spatially homogeneous.

2. Case of spatially uniform distribution

We start with a line element of the form

$$ds^2 = dt^2 - dr^2 - dz^2 + f\, d\phi^2 + 2m\, d\phi\, dt \tag{1}$$

which with

$$f = 4a^2(\sinh^4 x - \sinh^2 x) \tag{2}$$

$$m = 2a\sqrt{2}\sinh^2 x, \qquad x = \frac{r}{2a} \tag{3}$$

represents the Gödel solution.

We shall assume the coordinate system to be co-moving and that f and m are functions of r alone. Thus in the present group of solutions there will be neither shear nor expansion.

The electromagnetic field is assumed to be such that, except F^{13} and F^{14} (numbering r, z, ϕ, t as 1, 2, 3, 4), all other components of $F^{\mu\nu}$ vanish, i.e. an axial magnetic field and a radial electric field exist. It follows from Maxwell's equations $F^{3\alpha}{}_{;\alpha} = 0$ that

$$F^{13} = \frac{A}{(m^2 - f)^{1/2}} \tag{4}$$

A being a constant.

Further, since the Lorentz force $F^{\mu\nu}v_\nu$ vanishes everywhere

$$F^{14} = -\frac{Am}{(m^2 - f)^{1/2}}. \tag{5}$$

The other Maxwell equations $F_{[\alpha\beta,\mu]} = 0$ are automatically satisfied.

The Einstein equations are

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = -8\pi\rho v_\mu v_\nu - 8\pi E_{\mu\nu} + \lambda g_{\mu\nu} \tag{6}$$

where

$$E_{\mu\nu} = \frac{1}{4\pi}(\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - g_{\mu\beta}F^{\beta\alpha}F_{\nu\alpha}).$$

We note that λ is positive for the Gödel solution and negative for the Einstein universe. In view of (1), (4) and (5), the electromagnetic stress energy tensor $E_{\mu\nu}$ has the following non-vanishing components:

$$\left.\begin{aligned} E_{11} &= -E_{22} = E_{44} = \frac{A^2}{8\pi} \\ E_{33} &= \frac{A^2}{4\pi}(m^2 - \tfrac{1}{2}f) \\ E_{34} &= \frac{1}{8\pi}A^2 m \end{aligned}\right\}. \tag{7}$$

Using (6) and (7) one obtains after some calculation in the case $\lambda > 0$

$$m = a\,\mathrm{e}^{\alpha r} + b\,\mathrm{e}^{-\alpha r} + c \tag{8}$$

and

$$f = m^2 - \frac{\alpha^2(a\,\mathrm{e}^{\alpha r} - b\,\mathrm{e}^{-\alpha r})^2}{4(A^2 + \lambda)} \tag{9}$$

where $\alpha = (2\lambda)^{1/2}$ and a, b and c are constants of integration. Now, if the solution is to be cylindrically symmetric, ϕ is to be interpreted as an angular coordinate, and then for regularity at $r = 0$ one must have $f \to -r^2$ and m/r^2 regular as $r \to 0$ (Maitra 1966). This would require from (8) and (9)

$$a = b = -\tfrac{1}{2}c = \left(\frac{A^2 + \lambda}{4\lambda^2}\right)^{1/2}. \tag{10}$$

The solution can then be written

$$ds^2 = dt^2 - dr^2 - dz^2 + \left\{\beta \sinh^4(\tfrac{1}{2}\alpha r) - \frac{4}{\alpha^2}\sinh^2(\tfrac{1}{2}\alpha r)\right\} d\phi^2 + 8a\sinh^2(\tfrac{1}{2}\alpha r)\, d\phi\, dt \tag{11}$$

where

$$\beta = \frac{4A^2 + 2\lambda}{\lambda^2}, \qquad 8\pi\rho = 2(A^2 + \lambda) \quad \text{and} \quad 2\pi\sigma = A(A^2 + \lambda)^{1/2} \tag{12}$$

ρ and σ being the mass density and charge density, respectively. The line element (11) goes over to the Gödel form (equations (1), (2), (3)) when A vanishes.

However, unlike the Gödel case, we can have now $\lambda \leqslant 0$. For $\lambda = 0$ the solution becomes

$$ds^2 = dt^2 - dr^2 - dz^2 + (A^2r^4 - r^2)\, d\phi^2 + 2Ar^2\, d\phi\, dt \tag{13}$$

with

$$\rho = \frac{A^2}{4\pi} \quad \text{and} \quad \sigma = \frac{A^2}{2\pi} \tag{14}$$

where the constants of integration have been chosen to obtain a cylindrically symmetric system which is regular at $r = 0$ (ϕ being the angular coordinate). The solution (13) has been found already by Som and Raychaudhuri (1968). This solution is interesting as, unlike the other two stationary solutions of Einstein's field equations (Melvin 1964, Maitra 1966), where the energy distribution decreases as we go away from the axis, tending to vanish as $r \to \infty$, here the distribution of matter and electromagnetic energy is uniform.

For $\lambda < 0$ one obtains, again choosing the constants of integration suitably,

$$ds^2 = dt^2 - dr^2 - dz^2 + \left[h^2\{\cos(\gamma r) - 1\}^2 - \frac{\sin^2(\gamma r)}{\gamma^2}\right] d\phi^2 + 2h\{\cos(\gamma r) - 1\}\, d\phi\, dt \tag{15}$$

where

$$\gamma^2 = -2\lambda, \qquad h^2 = \frac{A^2 + \lambda}{\lambda^2}$$

$$8\pi\rho = 2(A^2 + \lambda) \quad \text{and} \quad 2\pi\sigma = A(A^2 + \lambda)^{1/2}.$$

As λ is negative we must have $A^2 > -\lambda$. It may be verified from the calculation of the scalar form of the Riemann–Christoffel curvature tensor that no real singularity occurs at values of r where g_{33} vanishes.

The solution (11) can be transformed as in the Gödel case to a form in which the spatial homogeneity is apparent. Thus the transformation

$$\left.\begin{aligned} \exp(x_1) &= \cosh(\alpha r) + \sinh(\alpha r)\cos\phi \\ x_2 \exp(x_1) &= \left\{\frac{2(A^2+\lambda)}{\lambda}\right\}^{1/2} \sin\phi \sinh(\alpha r) \\ \tan\left[\tfrac{1}{2}\phi + \frac{x_4 - \alpha t}{2\{2(A^2+\lambda)/\lambda\}^{1/2}}\right] &= e^{-\alpha r}\tan(\tfrac{1}{2}\phi) \\ x_3 &= \alpha z \end{aligned}\right\} \tag{16}$$

leads to the homogeneous form

$$ds^2 = \frac{1}{\alpha^2}\left\{dx_4{}^2 - dx_1{}^2 - dx_3{}^2 + \frac{A^2 + \frac{1}{2}\lambda}{A^2 + \lambda}\exp(2x_1)\, dx_2{}^2 + 2\exp(x_1)\, dx_2\, dx_4\right\}. \tag{17}$$

In the form (17) the solution has recently been given by Raval and Vaidya (1967) and also by Ozsvath (1967). Ozsvath, however, considered a fluid of infinite conductivity.

It is easy to note that all the three forms (11), (13) and (15) have closed time-like lines, those in (11) for

$$\sinh^2(\tfrac{1}{2}\alpha r) > \frac{4}{\beta\alpha^2} \tag{18}$$

and those in (13) for

$$r^2 > \frac{1}{A^2}. \tag{18$'$}$$

In (15) closed time-like lines occur in the regions defined by

$$\tan^2(\tfrac{1}{2}\alpha r) > \frac{1}{\alpha^2 h^2} \tag{19}$$

so that such lines occur in zones of r, separated from each other.

The above solutions, in which the cosmological term λ exists, can alternatively be interpreted as solutions with non-vanishing pressure p and vanishing λ. The field equations can be written in the form

$$-G_\nu{}^\mu = 8\pi(T_\nu{}^\mu + E_\nu{}^\mu) - \lambda\delta_\nu{}^\mu = 8\pi(T_\nu{}^{\mu\prime} + E_\nu{}^\mu) \tag{20}$$

where $T_\nu{}^{\mu\prime}$ is a new energy–stress tensor and $\lambda = 0$. Writing explicitly

$$8\pi\{(\rho' + p)v^\mu v_\nu - p\delta_\nu{}^\mu\} = 8\pi\rho v^\mu v_\nu - \lambda\delta_\nu{}^\mu$$

and comparing both sides

$$\rho' + p = \rho \quad \text{and} \quad 8\pi p = \lambda \tag{21}$$

one obtains a positive pressure with $\lambda > 0$, i.e. with the solution (11). Thus in this case one may write, using (12),

$$\rho' - p = \frac{A^2}{4\pi} \tag{22}$$

which shows that in general $\rho' > p$ and is therefore within the Zeldovich limit.

3. Case of non-uniform distribution

We shall next consider three cases, which correspond to non-uniform distributions of matter. Som and Raychaudhuri have discussed the case of a rigid rotation of cylindrically symmetric charged-dust distributions with vanishing Lorentz force. We shall in the first case retain this condition of vanishing Lorentz force but consider an axially symmetric distribution, i.e. the space–time no longer admits a translation along the axis of rotation. We consider the line element

$$ds^2 = dt^2 - e^{2\psi}(dr^2 + dz^2) - l\,d\phi^2 + 2m\,d\phi\,dt \tag{23}$$

where now ψ, l and m are the functions of r and z, and the dust velocity is given by $v^\mu = \delta_4{}^\mu$. Again, assuming that the only non-vanishing contravariant electromagnetic field tensor components are F^{13} and F^{14}, we may use Weyl-like canonical coordinates as in Som and Raychaudhuri's paper:

$$l + m^2 = r^2. \tag{24}$$

The Maxwell equations $F^{\alpha\beta}{}_{;\beta} = 0$ gives

$$F^{31} = \frac{A}{r}\,e^{-2\psi} \tag{25}$$

where A is a function at most of z.

Now, the vanishing of Lorentz force leads to

$$F^{14} = \frac{m}{r}A\,e^{-2\psi}. \tag{26}$$

It is not difficult to find from Einstein's field equations, using (24), (25) and (26), the following solutions:

$$m = rJ_1(kr)(C\,e^{kz} + D\,e^{-kz}) \tag{27}$$

$$l = r^2\{1 - J_1{}^2(kr)(C\,e^{kz} + D\,e^{-kz})^2\} \tag{28}$$

$$\psi = \tfrac{1}{2}A^2r^2 - \tfrac{1}{4}J_1(J_1 + krJ_1')(C^2\,e^{2kz} + D^2\,e^{-2kz}) - \tfrac{1}{2}CD\left(k^2\int J_1{}^2r\,dr + \int\frac{J_1{}^2}{r}\,dr + k^2\int J_1'^2r\,dr + J_1{}^2\right) \tag{29}$$

$$\rho = \frac{e^{-2\psi}}{8\pi}\left\{-2A^2 + \left(\frac{J_1{}^2}{r^2} + \frac{2k}{r}J_1J_1' + k^2J_1'^2\right)(C\,e^{kz} + D\,e^{-kz})^2 + k^2J_1{}^2(C\,e^{kz} - D\,e^{-kz})^2\right\} \tag{30}$$

$$\sigma = -\frac{A}{r} e^{-2\psi}(J_1 + kJ_1')(C e^{kz} + D e^{-kz}). \tag{31}$$

ψ is integrable only if A is a constant. C, D and k are constants of integration and

$$J_1'(x) = \frac{\partial}{\partial x}\{J_1(x)\}.$$

When the above solution extends over the whole range of z equation (27) shows that m increases indefinitely for large positive and negative values of z. Consequently $l\,(= r^2 - m^2)$ becomes negative when $m^2 > r^2$ and there occur time-like ϕ lines. Since ϕ is an angular coordinate the above ϕ lines are closed. Also from (29), for large values of r, ρ becomes negative if $A \neq 0$.

In the second case we consider a cylindrically symmetric system of uncharged-dust distributions, with a source-free magnetic field in which only F^{31} is non-vanishing. Then

$$F^{31} = \frac{A}{r} e^{-2\psi}, \qquad F^{41} = 0 \tag{32}$$

where A is a constant, so that by solving Einstein's field equations one can obtain solutions like

$$m = BrJ_1(2Ar) \tag{33}$$

$$l = r^2\{1 - B^2 J_1{}^2(2Ar)\} \tag{34}$$

$$\psi = \tfrac{1}{2}A^2r^2 - A^2B^2 \int rJ_1{}^2(2Ar)\,dr - \tfrac{1}{4}B^2 \int J_1{}^2(2Ar)\frac{dr}{r} \tag{35}$$

$$- A^2B^2 \int rJ_1'^2(2Ar)\,dr - \tfrac{1}{4}B^2J_1{}^2(2Ar)$$

$$\rho = \frac{e^{-2\psi}}{4\pi}\left\{J_1{}^2\left(\frac{B^2}{2r^2} - A^2B^2\right) + 2A^2B^2J_1'^2 + \frac{2AB^2}{r}J_1J_1' - A^2\right\} \tag{36}$$

where B is a constant of integration, and here ψ, m and l are functions of r alone.

Clearly, with a suitable choice of B, l would be positive everywhere, and there would be no closed time-like lines (cf. Maitra 1966). However, the expression for ρ shows that for large enough values of r it is negative. Hence these solutions cannot represent the distribution of matter with a positive density everywhere.

The last case that we shall consider is one in which, as in Maitra's universe, the shear is non-vanishing. Again, we assume an electromagnetic field with vanishing Lorentz force. Now, from field equations we can obtain easily the following relations:

$$4\pi\rho e^{2\psi} = \frac{m_1 m_{11}}{2r} - A^2 \tag{37}$$

$$m_{11} = \frac{m_1(1 - m_1{}^2)}{r(1 + m_1{}^2)} + \frac{4A^2rm_1}{1 + m_1{}^2}. \tag{38}$$

If there are no closed time-like lines, then $l\,(= r^2 - m^2)$ is positive, i.e. $m < r$ everywhere.

Hence, for $r \to \infty$, $m_1 \leqslant 1$ and

$$m_{11} \to \frac{4A^2rm_1}{1 + m_1{}^2}.$$

Integrating, we obtain

$$m_1 \exp(\tfrac{1}{2}m_1{}^2) = \exp(2A^2r^2).$$

Here $m_1 \to \infty$ as $r \to \infty$. This contradicts our initial assumptions that $m < r$. Hence this solution has closed time-like lines.

4. Conclusion

In the foregoing sections we have presented a number of solutions of Einstein's field equations for rotating matter with or without the λ term, and with a force-free electromagnetic field. It is remarkable that in all but one case there occur closed time-like lines. A point that seems to be of some interest in this connection is that the only stationary solution so far known, which is free of closed time-like lines, is the Maitra and the Melvin universe (Maitra 1966, Melvin 1964), in which, however, the matter (or the electromagnetic energy) is present in the infinite dilution. The solutions we have presented, as well as the solutions of Gödel (1949) and Van Stockum (1937), have, however, all finite average densities of matter and/or electromagnetic energy.

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